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ZENO AND THE MATHEMATICIANS

By G. E. L. OWEN.

AT some time in the first half of the fifth century B.C., Zeno invented the set of paradoxes on which his successors have sharpened their wits. The puzzles have come down to us in various versions, more or less incomplete and more or less reflecting the special interests of later writers. What we have left of Zeno's best-known work comes, on the most hopeful view, to less than two hundred words. Still, this has not stopped Zeno's admirers from trying, with all due caution, to reconstruct the programme of all or some of his arguments. I want to make one such programme plausible and to show how, if I am right, this makes some solutions to the puzzles beside the point.

My second interest in the paper is this. Zeno, it is commonly said, was and wished to be the benefactor of Greek mathematics. By his day the Pythagoreans had brought mathematics to a high level of sophistication. But the foundations of their system were a nest of confusions, and Zeno was out to expose these confusions. One beneficial result of his arguments (on this familiar account) was to compel mathematicians to distinguish arithmetic from geometry.

This picture seems to me mistaken. Zeno neither had nor tried to have this effect on mathematics (though in other ways, no doubt, he did influence contemporary work in the science). But his arguments had a great effect on a later stage of mathematics, and the effect was not beneficial.

Zeno's programme

Zeno certainly held, as a philosophical theory inherited from Parmenides, that there is only one thing in existence. This is an embarrassment to those who want to portray him as trying to set up a consistent logic for analysing the structure of space and time. For it means that he thought there was no such structure: any way of dividing things in time or space must carry absurdities. If this was his theory we should expect him to work out an exhaustive list of possible ways of dividing things and to set about refuting all the possibilities separately; and this, as I shall try to show, is what he does.

(Let me say at once that this talk of dividing is deliberately ambiguous. It is not always clear, for instance, whether Zeno is discussing the possibility of producing a plurality by actually carving a thing up or by enumerating the fractions it must logically contain; but for most of the way we shall find the distinction irrelevant.[1] What matters is that whichever operation Zeno has in mind he is canvassing its logical and not its physical possibility.)

Some hold that Zeno was not committed to any philosophical tenet whatever. For he is credited with saying " Show me what the *one* is and then I can tell you what things (in the plural) are ";[2] and this is sometimes taken to show that he did not profess to understand even the one thing that Parmenides had left in existence. But the point of his words is just that, if you want to say that there are a number of things in existence, you have to specify what sort of thing counts as a unit in the plurality.[3] If there can be no such individuals as you claim there can be no such plurality either. And in particular if your individuals have to be marked off by spatial and temporal distinctions you have to be sure that your way of making such distinctions is not logically absurd.

Plato makes it clear that Zeno's major work was divided into separate arguments, each depending on some hypothesis and reducing the hypothesis to absurdity. We do not know the content of these hypotheses, but Plato is emphatic that every argument was designed to refute the proposition that

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[1] It is not clear even that Zeno used the word διαιρεῖν and its cognates; but Parmenides had, and Zeno certainly used equivalent language in discussing Parmenides' topic (see A(1) below).

[2] Eudemus *apud* Simpl. *in Phys.* 97.12–13, 138.32–33.

[3] cf. Alexander *apud* Simpl. *in Phys.* 99.12–16.

there are a number of things in existence.[4] We certainly have reports of some of the arguments which began " Suppose many things exist ". But we also have a report of one which starts " Suppose place exists ". And Aristotle treats the familiar puzzles of *Achilles*, the *Arrow*, the *Stadium* and the *Dichotomy* as though these were designed in the first instance to refute the possibility of movement, not of plurality. It might be, of course, that these latter arguments came from another work of Zeno's.[5] But I shall try to show that they play an essential part in the attack on plurality.

Zeno's major question then is: if you say there are many things in existence how do you distinguish your individuals? The answer in which he is chiefly interested is that the world and any part of it can be broken down into its individual parts by spatial and temporal divisions. And the paradoxes that I am anxious to discuss are those designed to meet this answer, namely those which are jointly planned to show that no method of dividing anything into spatial or temporal parts can be described without absurdity.

For suppose we ask whether such a division could be (theoretically, at least) continued indefinitely: whether any division can be followed by a sub-division, and so on, through an infinite number of steps. Let us say, to begin with, (A) that it does have an infinite number of steps. Then could such a division nevertheless ever be (or ever have been) completed? (A1) One of Zeno's arguments is designed to show that it could not.

The paradox had two arms. The first began by arguing that the units in a collection can have no size at all: else they would have parts and be not units but collections of units.[6] The second began by arguing that, on the contrary,

[4] *Parmenides* 127e–128a, a version which became standard with later commentators (*e.g*. Simplicius *in Phys*. 139.5–7).

[5] Not that the evidence that he wrote other works is strong: Plato seems not to know of them, yet he certainly knew of the arguments on motion (cf. *Phaedrus* 261 d and the application of the *Arrow* in *Parmenides* 15 b-e).

[6] Simplicius, *op. cit*. 139.18–19: this argument at the start of the paradox is still overlooked by English editors, although its text and sense were settled by Hermann Fraenkel in the *American Journal of Philology*, 1942, 14–17 (= *Wege und Formen*, 211–214).

there cannot be anything that has no size at all; for there cannot be a thing which if it were added to or subtracted from something else would not affect the size of that thing.[7] So the first arm of the argument assumes that the units it describes are theoretically indivisible; and the point of this requirement comes out in the sequel, when Zeno shows that he is discussing the class of individuals produced by an *exhaustive* division of something, a division whose end-products cannot themselves be further divided. The second arm of the argument assumes, on the other hand, that its units must be capable of being added and subtracted in a sense in which these operations cannot apply to things without magnitude; and the point of this requirement comes out in the same sequel, for if a thing can be divided into parts (exhaustively or not) those parts must be capable of being added to make the thing, and in that case they must have some size, however small.

Next, to bring these requirements into one focus, Zeno went on to specify the collection of parts in which he was interested, namely the collection produced by completing a division in which every step has a successor. " Each thing ", he said, " must have some size and thickness, and one part of it must be separate [or perhaps just ' distinct '] from another. And the same holds good of the part which is in the lead—that too will have some size, and of it too some part will be in the lead. In fact to say this once is as good as saying it for ever, for no such part of the thing will be the last or unrelated to a further part."[8] These words define a division so that there can be no last move in the sequence: for any fraction that is taken, a similar fraction can be taken of the remainder (the " part in the

[7] Simplicius, *op. cit.* 139.9–15.

[8] Simplicius, *op. cit.* 141.1–6. A commoner but linguistically less easy version of the words runs " Each thing must have some size and thickness and there must be another thing separate from it. And the same holds good of the thing in front: it too will have some size and there will be something in front of it . . ." Taken in this way the words do not define the steps in the division but merely characterize its products by saying that the series has no last member. And there is no mention of parts (more exactly, none of the Greek genitives is understood as partitive) before the last line. Otherwise, for our purposes both versions come to the same.

lead "). In this, certainly, there is no clear implication that such a division can have been completed. But Zeno does make that assumption in drawing his conclusions. For he points out that, on one line of argument (that of the first arm), the parts produced by this division can have no size at all: they are end-products whose further division is logically impossible. And he also points out that, on the other line of argument (that of the second arm), since all the parts of such a collection must have some size the whole collection (and by the same token any part of it) must be infinite in size. And both conclusions are absurd. They were presented as an antinomy; but as a dilemma they are equally lethal. Either the parts have no size, and then there can be no such parts; or they have some size, and then the thing you set out to divide becomes infinitely big.

Notice that Zeno is not first setting up a division which cannot have a last move and then asking, improperly, what the last move would be.[9] He is asking, legitimately, what the total outcome of the division would be; and for there to be such an outcome there must be a smallest part or parts.

The effect of the argument is to show an absurdity in the alternative for which we opted first, namely that if anything is infinitely divisible such a division can be carried right through. So now (A2) we shall say that anything is infinitely divisible but that such divisions can never be completed. Then, supposing that the puzzle about Achilles and the tortoise[10] is a puzzle about infinite divisibility, it is designed to block this escape-route. In order to overtake the tortoise Achilles must first reach the tortoise's starting-point; but by then the tortoise will have reached some further point. So then Achilles must reach this point, by which time the tortoise will have got on to another, and so forth: the series comes to no end. The moves which Achilles is required to make correspond to divisions of the intervening country, and the divisions are infinite, determined by the same general formula as in A1. But on our present assump-

[9] cf. J. F. Thomson, 'Tasks and Super-tasks'; *Analysis* xv (1954–5), 6–7.
[10] Aristotle, *Physics* Z 239b14–29.

tion Achilles cannot complete any such sequence of moves; so he cannot overtake the tortoise, whatever their relative speeds and however short the lead.

Now if I am right about the coupling of Zeno's arguments[11] it is beside the point to maintain, as a general solution of this puzzle, that an infinite division can be completed. For if we say this Zeno will take us back to A1 and ask us about the character of the ultimate parts produced by the division. To make this clear, consider Aristotle's first solution to the puzzle—a solution which he later admits to be unsatisfactory but which he nevertheless thinks to be adequate *ad hominem*.[12] He replies that, provided we recognize that the time of the run can be divided in just the same way as the ground, Achilles can overtake the tortoise in a finite time; for the smaller his moves become the less time he needs to accomplish them, and these component times can diminish without limit. Then suppose we tell Achilles to mark in some way the end of each stage of the course in which he arrives at a point reached by the tortoise in the previous stage. Suppose also we satisfy Aristotle's requirement and allow these successive markings to follow each other at a speed which increases indefinitely, in inverse ratio to the ground covered at each stage; and suppose the marks become proportionately thinner and thinner. Then Zeno, as I understand him, argues that if Achilles claims to have finished his task we can ask about the positions of these marks, and in particular of the last two. If they are in the same place there is no stage determined by them, and if there is any distance between them, however small, this distance is the smallest stage in an infinite set of diminishing stages and therefore the course is infinitely long and not just infinitely divisible.

Two things, I take it, we must give Zeno: first, that of the series of movements that Achilles is supposed to make there can be no last member, just as of the stages of the division

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[11] Notice the προὒχειν which may have been common to both puzzles: Simplicius *op. cit.* 141.4; Aristotle, *op. cit.* 239b17.

[12] *op. cit.* 233a21–31, 263a15–18: the solution is applied first to the *Dichotomy*, discussed below, but Aristotle took this to be the same puzzle as the *Achilles*.

described in A1 there can be no last stage; and second, that if either series can be completed it must be possible to describe the resulting state of affairs without absurdity. From these admissions Zeno infers that Achilles can never finish the run that brings him level with the tortoise. Any hope of salvation lies in looking at this inference.

Consider that other series of moves to which Professor Black once likened Achilles' run.[13] Hercules is required to cut off the Hydra's heads, but every time he cuts off a head another grows in its place. When can he finish the assignment ? Never, if the task was correctly specified. For if some heads are left on Hercules has more work to do, and if all are off it was not the case that for every head cut off another head grows. But these are exhaustive alternatives, so there is *no* subsequent state of affairs of which it is logically possible to say truly: Hercules has finished his task. Now (as Mr. Watling has already argued)[14] this is not the case with Achilles. There are plenty of states of affairs compatible with Achilles' having achieved his task of overtaking the tortoise: plenty of positions beside or beyond the tortoise that Achilles can have reached. It is just the case here that Achilles' movements have been so described that they have no last term, but not so that no subsequent state of affairs is compatible with his having completed the series. But to require anyone to finish an infinite division, as in A1, is to start them on a Hydra-operation: there can be no state of affairs, no collection of bits, of which it is possible to say: Now the job is done. For either the bits do or they do not have some size, and that exhausts the subsequent possibilities. On this Zeno was right. His error was to construe his A2 example on the model of A1.

In a later paper[15] Black admits this difference in the sense in which Hercules and Achilles can be said to have taken on an infinite set of tasks. But he still holds that in either case " talk of an infinite series of acts performed in a finite time is illegitimate ". For he now says that the description of

[13] *Analysis*, xi (1950–1), 98 (=*Problems of Analysis*, 105).

[14] *Analysis*, xiii (1952–3), 41–2.

[15] " Is Achilles still running ?" *Problems of Analysis*, 109–126.

Achilles' movements belongs to " common-sense language " which, in contrast to the mathematical representation of space and time, " does not permit talk of the indefinitely small "—that is, does not have a use for describing Achilles' movements as becoming as short as you like. But a guillotine is not an argument. If someone says, " In making any movement you make an infinite series of decreasing movements ", we have no reason yet to reject this as an offence to common usage. It already looks like a recognizable application of mathematical language to the description of familiar events (we recall the graphic problems in school arithmetic): what it needs at once is clarification. We can ask " What do you mean here by ' infinite series '? Do you say that in walking from *a* to *d* I make a set of smaller walks of which the first takes me beyond *a* and the last brings me level with *d*? For then I cannot see how you define this sequence so as to let me draw on my knowledge of other uses of ' infinite ' ". Suppose then he gives us a formula, as in A1 or A2, for defining the class of movements so that there can be no last move in the sequence bringing me level with *d*. Then we know how he is using the redescription of our movements that he has introduced. He has not uncovered an unsuspected set of events in our daily histories and he has not burdened us or Achilles with a new and crippling set of duties: the connexion between our usual descriptions of Achilles' run and this sort of restatement is not in either of these ways a factual connexion. What we have been given is a translation of those usual descriptions; where the second can be known, directly, to apply, the first can be known, derivatively, to apply. So no consequential question can arise about the applicability of the second. If we are told that the equation shows why Achilles never can catch the tortoise, we can only complain that the proffered rules of translation have broken down and go back to our request for clarification. Any attempt at this stage to reconstrue the expression " series of moves with no last member " as specifying a Hydra-operation, an infinite parcelling of the ground such that no state of affairs is compatible with its completion, cancels the equation with our description of Achilles' run. And in this way " common-sense language " is safeguarded; for it is the oscillation between bringing in the infinite series as a logically innocuous translation of ordinary statements and trying to reconstrue it on the model of the task in A1 that breeds the puzzle.

A closely associated paradox is the *Dichotomy*.[16] Before reaching your destination you must reach half-way, but before reaching that you must reach half-way to it; and so back. So in this series there is no first move, and you cannot get started. (It can of course also be made to show that there is no last move; Aristotle's report here is ambiguous. But this was taken care of by the *Achilles*.) Here again if you insist that there is a first move you are taken back to A1: either this move is no move, or it covers some distance, however small. The solution here is the same as for the Achilles. But Aristotle says that in face of this puzzle some theorists (certainly Xenocrates and apparently at one time Plato) postulated atomic distances, " indivisible lines ".[17] That is, they challenged Zeno's disjunction " Either no size at all; or some size, and then divisible " by adding " Or some size, but *not* divisible ". Then the first or last move towards one's destination would be to cover such an atomic distance; for one could not logically be required to cover any fraction of it first. It is not certain whether the proponents of this theory thought that any measurable distance contained a finite or an infinite number of such distances. An argument for thinking that they meant the former is that this is assumed in the fourth-century polemic *On Indivisible Lines*. An argument for thinking the contrary is that the theory was held at a time when the difficulties of incommensurable lines were fully realized. It was a commonplace that the side and diagonal of a square cannot both be *finite* multiples of any unit of length whatever. If the latter account is true, those who introduced this theory were suggesting that an infinite division can have a last

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[16] Aristotle, *Physics* Z 239b11–14 (cf. 233a21–23).

[17] *Physics* A 187a1–3.

term: the products of such a division are not completely without magnitude, yet they have no finite magnitude such that fractions of it can be specified. They would be, in fact, to all present intents and purposes, infinitesimals, vanishingly small quantities; and movement over such a distance is what writers on mechanics such as Heinrich Hertz have called *infinitely small* or *minimum displacements.* But whichever interpretation of the theory we give, it was an attempt to evade Zeno's dilemma in A1.

Now it looks as though this attempt is met in advance by another of Zeno's arguments, that known as the *Stadium.*[18] On the prevalent interpretation of the argument this is certainly so; and I wish I could be sure of the truth of the interpretation. But it is fair to warn you that, if the moral of the argument is anything like that now found in it, the Greeks seem to have missed the point by a wide margin. Plato, who converted many of Zeno's arguments to his own use, made no use of this one and apparently saw no objection to postulating infinitesimals. Aristotle rejected infinitesimals, but he missed the sense of an argument that Plato had missed before him.

The puzzle sets up three parallel rows of bodies. All the bodies are equal in size; each row contains an equal number of them; and (a stipulation omitted in Aristotle's report) the bodies in each row are directly adjacent. One row (the *A*s) is stationary. The other two (*B*s and *C*s) meet at the mid-point of the *A*s and move on past each other at equal speeds, so that when the first *B* clears the last *A* in one direction the first *C*, moving in the opposite direction, clears the last *A* at the other end. Thus in the time that the first *B* passes half the *A*s, from mid-point to end, it passes all the *C*s. Let this time be t. But then if the first *B* takes t to pass n bodies (to wit, half the *A*s) it must take not t but $2t$ to pass $2n$ bodies (*viz.* all the *C*s). So the move which takes t also takes $2t$; this is the alleged puzzle, and plainly it depends on disregarding the relative motions of the bodies. The *C*s are moving, the *A*s are not. That is Aristotle's sole comment on the argument, and it is generally

[18] Aristotle, *Physics* Z 239b33–240a18.

felt that if it is refuted by such a comment it was not worth the considerable space he gave it.

Suppose now that Zeno asks how we can specify the relative motions of the bodies. If we say that the first *B* can pass twice as many *C*s as *A*s in a given time, what we say entails that if in a given time it passes one *C* it also passes half an *A*. But suppose now that any *A* (and therefore any *B* or *C*) is an *infinitesimal* quantity. Then the *B* cannot pass half an *A*: it must pass all or nothing. And since *ex hypothesi* it *is* moving past the *A*s it must pass a whole *A* in the time that it passes one *C*. Yet, as we set up the problem, it would pass twice as many *C*s as *A*s in a given time. So when it passes one *C* it also passes two *C*s, and this gives Zeno his contradiction. It seems the simplest hypothesis that gives the problem any weight whatever.[19]

There is a familiar argument to show that, if lengths are made up of infinitesimal lengths, everything that moves must move at the same speed.[20] Zeno goes one better than this. He argues (on the present interpretation at least) that, if bodies are made up of infinitesimal lengths, then even if bodies do move at the same speed they cannot move in opposite directions.

This argument, then, seems designed to destroy the last hope that the sort of division described in A1 could theoretically be terminated, in the sense of producing any specifiable end-products. And the *Achilles* and the *Dichotomy* were devised to eliminate the alternative, that the world or any part of it was open to an infinite division that did not terminate in any end-product. The next question is whether Zeno faced the alternative (B) that any division terminates in some finite number of steps beyond which no further step is even logically possible.

Against one arm of this option he did not, so far as we know, think it worth arguing, namely the joint assertion

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[19] But it is possible that Zeno was out to explode the distinction between *moving* and *static*. Given that the distinction is relative, any one of the rows of bodies could be taken as providing the units of distance for assessing the speeds of the others. Trading on the fact that no row had prime right to this status, Zeno gave it to two of the rows in the same argument.

[20] Cf. Russell, *Principles of Mathematics*, §322.

that (a) anything is divisible for only a finite number of steps and (b) the products of such a division will have some finite size. For this could only be a thesis about physical possibilities: Zeno assumes without argument in A1 that the conjunction of size with theoretical indivisibility would be a contradiction. Suppose, on the other hand, that the products of such a division are said to have no size: then the argument of A1 that all parts must have some magnitude goes home against this thesis too. And suppose it is said that the products are vanishingly small, then the *Stadium* argument is equally effective here. For neither of these arguments requires that the end-products with which it deals should be produced by an infinite rather than a finite number of divisions.

However, I think that another of Zeno's arguments may be levelled directly against option B.[21] This is the argument that a collection containing a finite number of parts must also contain an infinite number of them. It must contain just the number that it does, whatever that number is; but between any two members there must be another member, so that the collection is infinitely numerous. The writer who reports this argument takes it to be concerned once again with the results of an infinite division.[22] But it can be understood more generally, as a foretaste of Bradley's paradox. Any two members of a collection must be separated by something if they are to be two things and not one; but by the same argument what separates them must itself be separated from each by something else; and so forth. I suspect that this is the correct interpretation because the argument then becomes complementary to one of Parmenides'. Parmenides had urged that if two things are separated it must be by a gap, nothing; but this is to mistreat nothing as a substantial part of the world.[23] Zeno reinforces this by extracting a different embarrassment from the plea that things are separated not by nothing but by other intervening things, substantial parts of the world.

[21] Simplicius, *op. cit.* 140.28–33.
[22] *Ibid.* 140.34–5: Simplicius on his own authority ?
[23] Diels-Kranz, *Vorsokratiker*[6] 28 B 8: 22, 46.

And his argument begins from the consideration of a finite collection; so it may well be aimed at any who thought that there must be some finite number *n* such that the world could be divided into *n* things but not—logically not—into any number higher than *n*.

Certainly, this argument seems patently fallacious. For surely things may be separated by their common boundaries —by their edges, and nothing else. And it is absurd to ask what separates them from their edges, absurd for the reason that Plato and Aristotle drove home, that the edge of a thing is not another thing of the same type as what it borders, not a part that can be cut off its possessor. The moment that begins a stretch of time or the point that bounds a line is not any stretch, however small, of time or space. Otherwise it in turn has a beginning, and then Zeno's regress is afoot. And Zeno is accused of ignoring this distinction.

If that is so we can turn to the argument through which, if through any, Zeno exercised a major influence on the mathematics of science. For it is in this argument above all that he is accused of confusing edges with the things they border, or more precisely of confusing instants, which are the limits of time-stretches, with time-stretches. But it seems equally likely that he is now characteristically trying to seal off an escape-route from the last argument by showing how absurdities come from the attempt to *distinguish* moments from periods of time. This remaining puzzle is that known as the *Flying Arrow*. But before discussing it let me bring the mathematicians into the picture.

The mathematicians

Most handbooks written since the time of Paul Tannery will tell you the purpose of the arguments we have examined so far. By Zeno's day Greek mathematics, in the hands of the Pythagoreans, had come to exhibit the familiar picture of a sophisticated superstructure built on badly confused foundations. In his arguments on divisibility Zeno was out to expose these radical confusions, and he succeeded.

Following some other writers I am inclined to think this explanation a myth, and an obstructive myth. For first, the picture of Pythagorean mathematics, to the extent that it is intelligible, rests on quite inadequate evidence. And secondly (and for the present paper more relevantly), if there were such a stage in the history of mathematics, Zeno's arguments would not be directed primarily at it.

Briefly, the theory ascribed to Zeno's contemporaries is this. It is mainly the work of Paul Tannery, but later writers have added to it. Cornford, one of the most important of these, credits the Pythagoreans with failing to distinguish physical bodies from geometrical solids, and with holding about these solids *both* that they are infinitely divisible *and* that they are divisible into atomic bits, which bits *both* have magnitude *and* have the properties of points without magnitude.[24] Indeed they seem to have held every possible opinion about the divisibility of bodies save the opinion that bodies are not divisible. Certainly, Zeno was anxious to find confusions in the claim that bodies are divisible at all. But to ensure that he was writing with a special target in view the target has been enlarged to the point where a shot in any direction will hit it.

This is not the place to hold an autopsy on the evidence for this theory. Much of the work has been done in print,[25] and what needs to be added can be deferred. What is to our purpose is that Zeno's arguments cannot have been directed against such a theory unless his whole programme was misconceived. For in order to provide his arguments with a target a theory had to be produced which housed every or nearly every incompatible view on the divisibility of bodies. But the direct refutation of such a theory would be to show the absurdity of holding any two or more of these views concurrently. What Zeno does is to distinguish each view and refute it in isolation. Thus he deals

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[24] Tannery, *Pour l'Histoire de la Science Hellène*, ch. x; Cornford, *Plato and Parmenides*, 58–9, and papers in the *Classical Quarterly*, 1922–3.

[25] In particular by Calogero and Heidel, van der Waerden, Fraenkel and Vlastos.

separately with absurdities arising from the addition of numbers (in the argument just discussed) and from the addition of magnitudes (in A1), although for Tannery the basic confusion in Pythagoreanism was the confusion between numbers and magnitudes. And he wrings separate embarrassments from the option that the ultimate parts of things have no magnitude and the alternative option that they have some magnitude, and again from the possibilities that a continuous dichotomy can and cannot be completed. In brief, his arguments seem designed to close not some but all avenues of escape to anyone holding the unremarkable belief that there is more than one thing in existence. To suppose that he is merely attacking the possibility of taking more than one of these avenues at once is to wreck the structure of his arguments and to neglect such evidence, internal and external, as we have of their motivation.

Now let me reset the scene by reminding you of some real teething-troubles that had overtaken mathematics by the time of Plato and Aristotle. The early Pythagoreans had certainly worked on the assumption that any two lengths can be represented as related to each other by a ratio of whole numbers. Any geometrical theorem could be applied in terms of the theory of numerical proportion that they had developed on this basis. But before Plato's day this assumption had run up against the discovery that lines could be constructed which bore no such proportion to each other. No matter what positive integer is assigned to the side of a square, no corresponding integer can be found to represent its diagonal.[26]

Some text-books would let you suppose that this discovery compelled mathematicians to jettison the old theory of proportion. But several reactions to it were possible. One was to retain the theory but restrict its scope: and this is just what Euclid does with it in the seventh Book of his *Elements*. One was to retain it *and* apply it to the sides and diagonals of squares by an accommodation that could be

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[26] Cf. Euclid, *Elements* X app. xxvii (Heiberg).

made as small as you please.[27] And one was the reaction of Eudoxus: to remodel the theory radically by allowing the concepts of *addition* and *greater* and *less* to range over rationals and irrationals alike.

This is enough to certify that the discovery of incommensurables was a real crisis in mathematics, and to introduce another type of reaction to it. Some mathematicians gave up the model of a line as a multiple of unit parts, a model which made sense only on the old theory of proportion. They said instead, as Newton said later,[28] that a line should be considered as generated not by the summation of parts but by the fluxion or motion of a point: the extended line is the path of a moving thing without extension. This is said to be a relatively late reaction,[29] but it is already under attack in Plato's *Parmenides* and Aristotle's *Physics*; and this attack ushered in the period of Zeno's most powerful influence on mathematics. So far, it is plain, Zeno has made no appearance in the crisis. Some writers, hoping to find for him a directly influential rôle in the mathematics of his day, have suggested that the new picture of a line as the path of a moving point was a response not to the discovery of incommensurable lines but to the arguments of Zeono; but this seems incredible. No one who had been vexed by those paradoxes can have hoped to evade them by introducing the idea of motion. In fact it is by an adaptation of some of Zeno's arguments that Plato rejects the new picture of a line; but Zeno himself had probably not talked of points and lines, and the later and precise concept of a point as something with location but without magnitude

[27] An ingenious but infertile device. A rule was devised for constructing a series of fractions approaching as close as you please to the ratio between side and diagonal: the lines were described by a series of paired numbers such that always the square on the diagonal equalled twice the square on the side plus or minus one, and the approximate sides and diagonals defined by this construction were called the " rational " sides and diagonals (Theon of Smyrna, 42.10–44.17 (Hiller), cf. Plato, *Rep.* 546c and Proclus' Commentary, ii.27 (Kroll)).

[28] *Quadr. Curv.* (1704), intro. § 27.

[29] Sextus Empiricus, *adv. math.* X 281–2. The concept of a line that was superseded by the fluxion-model is probably not the innocuous one compared with it by Sextus (279–280) and Proclus (*in Eucl.* i 97–8).

seems to have been produced to meet a difficulty that had little or nothing to do with his work.

When Plato turns to attack this account of a point in the *Parmenides*, he argues that a thing without parts cannot have a location.[30] For to have a location is to have surroundings, and this is to be in contact with something on various sides at various points: but a thing without parts cannot have different sides or points. This equation of location with surroundings is standard with the Greeks: Zeno had built one paradox on it,[31] and Aristotle was to give his own sophisticated version of it in the fourth Book of the *Physics*. Until it was replaced by the method of fixing location by co-ordinates, the formal objection to allowing a point location went unanswered. Aristotle inherited it,[32] as he inherited the corollary argument that a point cannot be said to move.[33] Moreover when Plato goes on to define the conditions under which anything *can* be in contact with different things and, in particular, can be a member of a linear series of such things, he provides both the pattern and the terminology for Aristotle's own treatment of points and lines in the *Physics*.[34] Aristotle's insistence that a line can be composed only of smaller, indefinitely divisible lines and not of points without magnitude rests on Plato's treatment of the point as a thing that cannot have sides or neighbours; and it is more than likely that Plato's argument derives from Zeno's warning that the parts of anything must have some magnitude, however small.

Now it is this same distinction between lines and points that Aristotle turns against Zeno's remaining puzzle, the *Flying Arrow;* and his mishandling of both the distinction and the puzzle is the last topic I want to discuss.

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[30] *Parm.* 138a: part of the attack described by Aristotle in *Metaphysics* A 992a20–22.

[31] Diels-Kranz, *op. cit.* 29 B 5.

[32] *Physics* 212b24–25.

[33] *Parm.* 138c-d; *Phys.* 240b8–241a6.

[34] Terminology: *contact* (ἅπτεσθαι), *in succession* (ἐφεξῆς), *neighbouring* (ἔχεσθαι), *Parm.* 148e and *Phys.* 226b18 ff. Plato defines the first by means of the other two, Aristotle defines the last by the first two.

The Arrow

Zeno's last paradox concerning motion is given by Aristotle in a form which, despite the depravity of the text, can be articulated as follows: Anything which occupies a space just its own size is stationary. But in each moment of its flight an arrow can only occupy a space just its own size. Hence at each moment of its flight the arrow is not moving but stationary. But what is true of the arrow at each moment of a period is true of it throughout the period. Hence during the whole time of its flight the arrow is not moving but stationary.[35]

Aristotle says that the fallacy lies in assuming that any stretch of time is a collection of moments, a mistake parallel to thinking that any line is a collection of points. Now in a sense his diagnosis is right; but not in the sense that he gave to it. Before we come to this, however, one small point needs to be made. Aristotle is often represented as accusing Zeno of thinking that any time-stretch consists of a *finite* collection of moments. But we shall see that Zeno does not need this premiss (nor its denial, either). And as for Aristotle, he was equally anxious to deny that a period could be composed either of a finite or of an infinite number of moments. Define moments as having no magnitude, and Aristotle has learnt from Zeno to argue that no magnitude can be in either of these ways a sum of such parts.

Let us clear some issues by an imaginary conversation.

Aristotle: You claim that (a) in each moment of its flight the arrow must be stationary, since evidently it has no time to move; but (b) what is true of it at each *moment* is true of it throughout the whole *period.* Hence your conclusion. But you agree that moments have no magnitude (that, of course, is why the arrow cannot move in one). Consequently they cannot be added together to make a period of time, which does have a magnitude.

Zeno: You seem to be attacking my premiss (b). I grant what you say: indeed my argument depends on stressing this characteristic of points and moments. (You

[35] Aristotle, *Physics* Z 239b5–9, 30–33: on the text cf. Lee, *Zeno of Elea* 78–81.

remember that I was accused of overlooking it last time.) But the argument does not require that the moments should be added together. I merely assumed that if something was true at any and every moment of a period it was true throughout a period. It is ordinary sense and not bad logic to say that if at any moment this afternoon I was asleep—at 4.30 as well as 2, and at any such precise time you care to take—then I was asleep thoughout the afternoon.

Aristotle: But you cannot describe periods *exhaustively* in this way, in terms of moments. However many moments you can mention you are still only specifying the limits of the periods that separate them, and at any stage of the division you like it is these periods that make up the overall period. You can never have two neighbouring moments. So if it is correct to infer from the fact that at any time this afternoon I was asleep, to the fact that I was asleep all afternoon this can only be because " at any time " means " at all periods, however small ". And " at 4.30 " can only mean, in this context, " at some period however small round 4.30 ". Don't misunderstand me: I am not suggesting that such time-references as " 4.30 " are really specifications of periods of time: if they were, we should have to invent a new set of time-references to say when such periods began and ended; and it is absurd to ask how long 4.30 lasts. I am only suggesting that *here* what parades as a time-reference must be a shorthand specification for some small period of time.

Zeno: In as far as this argument differs from your first, it is trifling. To specify moments is surely enough to specify the limits of periods. But to say that therefore any formula phrased in terms of moments is indirectly about periods of time merely invites the converse reply: for to identify a period is to describe the moments that define it.

Aristotle: Nevertheless you do talk about moments in a way that is only appropriate to periods. You say, for instance, that the arrow is stationary at every moment of its flight. But in the section of my περὶ κινήσεως which introduces an attack on your paradoxes[36] I show that if there is no time in a moment for the arrow to move there is no time for it to be stationary either. Movement involves having different positions at different moments, and accordingly rest involves having the same position at different moments. But we are considering only one moment, so neither of these ideas applies. In making either of them apply you treat the single moment as a period of time itself containing different moments.

Zeno: Now, in effect, you are turning your attack to my premiss (a). But if it is true that at any moment of its flight the arrow is neither moving nor at rest then, by my second premiss, the arrow is throughout its flight neither moving nor at rest. And as a paradox that will do—unless you can find some independent argument against my second premiss. Of course, if that premiss also depended on treating moments as small periods, the argument would collapse. But you have not shown this so far.

Aristotle: It might be shown like this. Consider a spatial analogy to your argument about time. If a surface is uniformly red all over it is red in every part of it, however small the part. But it is not red or any other colour at every point, if by " point " you mean something without extension. In the ordinary sense of "red" we have no use for calling something without extension red. If we had such a use it must be because " red " was used here in an unfamiliar sense. Likewise, even if it were legitimate to infer from " The arrow was moving (or at rest, or neither) at each moment " to " The arrow was moving (or at rest, or neither) throughout the period ", this could only show that the expression " moving " (or the expression " at rest ", or both) was being used ambiguously between the two cases. Your second premiss, if it is true, rests on a pun; but if it rests on a pun the conclusion you want will not follow.

Zeno (by now a prey to sharp anachronism): This is surely wrong. For suppose a body is constantly increasing its speed: this state of affairs is naturally explained by saying that *at any moment* it moves at a speed greater than at any

[36] *Physics* Z 239a23–239b4.

previous moment since its motion began. And here notice that the verb " to move " is associated with the common expressions for velocity and that it can be paraphrased by the common equivalents, " to change position " and so forth. So it is false that, if " motion at an instant " had any use, " motion " would have a different sense here from that which it usually carries.

Arbiter: You are both right and both wrong. Consider again the expressions " X was moving at some moment *t* ", " X was moving throughout the period *p* ". Aristotle denied that the expression " X was moving " had the same sense in both contexts. And in face of Zeno's reply we can add that any expansion of the expression, such as " X was moving at velocity V", could not have the same sense in both. For consider how the methods of confirmation differ. Velocity is distance measured against time. The simple question, With what velocity did X traverse *d* in the period *p* ? gets the simple answer, *d*/*p*. But the question, With what velocity was X moving at a time *t* inside that period ? is complex. It calls for the concept of a limit—the possibility of measuring an indefinitely long series of distances against a corresponding series of times. It can be answered, for instance, by constructing a graph whose curve is indefinitely corrigible by further pairs of measurements. To be sure, once we have this graph we can replace our simple question about speed over a period with a more sophisticated one. For whereas our first question merely demanded the overall speed (not the *average* speed: this is again complex), we can ask now whether X's speed over the period was constant. And this involves a different use of the graph. To say that X moved with a constant speed during the period is to say something doubly general, when to ascribe it that speed at one moment is to say something singly general: for now we ascribe it a speed at each moment in the period. But the possibility of operating on either of these levels of generality depends on being able to answer questions of our first, simple form, and the converse is not true. And thus Zeno's rejoinder fails. For since in this way the possibility of talking about motion at a moment rests on the possibility of talking of motion over a period, the two uses of " motion " are not the same. Likewise we could if we wished give a use to the expression " colour at a point " by building on our ways of describing a colour over a space, but we could not begin the other way round without a radical change in the use of colour-words. But in another way Zeno was right. For to say that these are not the same use is not at all to say that Zeno's second premiss depends on a pun. The premiss is valid, and it is valid precisely because it is the sort of rule whereby we do give a use to such an expression as " moving at a moment ". We rule that, when and only when it is correct to say " X was moving throughout the period *p* ", it is also correct to say " X was moving at any moment *t* in *p* ". Aristotle's fallacy lay in supposing that to infer from the second formula to the first, one must regard the second as specifying a conjunction of moments exactly as long as the period specified in the first. He was in fact applying a simple model of induction, that model which set a premium on the exhaustive enumeration of cases and which Aristotle took to require strict synonymy between different occurrences of the predicate (" X-moving ", for instance, in the inference from " Each moment in *p* is a case of X-moving " to " *p* is a case of X-moving "). And thus he failed to grasp that the two senses of " moving " are not identical but yet systematically connected; and his failure to see this connexion between two common uses of a common word led him to rule out one use entirely in favour of the other. His reply to Zeno rejects all uses of " movement " other than that which can be described in terms of periods of time, just as the colour-model we considered exhibited all uses of " red " as applicable to colour-stretches. And this is an unjustified departure from usage: it deprives us of a convenient method of characterizing motion which is common idiom for us and for the Greeks.

Now (and here we can drop the pretence of dialogue) if this is so Zeno's fallacy cannot lie in his second premiss. Therefore it lies in premiss (a), and in particular in the proposition " There is no time to move in a moment " (with or without Aristotle's rider: "and no time to rest either"). The picture we are given is of the arrow bottled up in a piece of time that fits it too closely to allow any movement. The moment is too short to fly in. But such talk of movement is appropriate only when we have in mind periods of time within which movements could be achieved. It is not false that movements can be achieved *within* moments: it is absurd either to say or to deny this, for moments are not pieces of time such that within them any process can either take place or lack the time to take place. But this certainly does not show that the arrow is not moving *at* any moment. It is, of course: we have seen the sense in which it is. Whether it is, is a question of fact and not of logic.

So, despite his contrast between moments and periods of time, Zeno was treating moments as stiflingly small periods. To that extent Aristotle was right in his diagnosis. But he did not apply the diagnosis where it was needed His denial that there can be any talk of motion except in direct connexion with periods of time is a surrender to Zeno; and his failure to come to grips with premiss (a) compels him to struggle against the wholly respectable premiss (b).

This surrender to Zeno had notable results in the history of dynamics. Notoriously, Aristotelian dynamics failed to deal adequately with acceleration; and it might be thought from what has been said that the failure lay in insisting that acceleration (a phenomenon which Aristotle certainly took seriously) must be analysed in terms of motion and speeds over periods of time, and not in the more manageable shorthand of velocity at an instant. But this is not the root-issue. Unable to talk of speed at an instant, Aristotle has no room in his system for any such concept as that of initial velocity or, what is equally important, of the force required to start a body moving. Since he cannot recognize a moment in which the body first moves, his idea of force is restricted to the causing of motions that are completed in a given period of time. And, since he cannot consider any motion as caused by an initial application of force, he does not entertain the Newtonian corollary of this, that if some force F is sufficient to start a motion the continued application of F must produce not just the continuance of the motion but a constant change in it, namely acceleration.[37] It is the clumsy tools of Aristotelian dynamics, if I am right, that mark Zeno's major influence on the mathematics of science.

[37] He would have had another reason for rejecting Newton's account of acceleration, for that account holds good only in a vacuum, and Aristotle thought a vacuum impossible. But some of his followers re-imported the vacuum without abandoning the rest of the system.